

Precision Primordial ${}^4\text{He}$ measurement from the CMB

Greg Huey,^{1,*} Richard H. Cyburt,^{1,†} and Benjamin D. Wandelt^{1, 2,‡}

¹*Department of Physics, University of Illinois, Urbana, IL 61801*

²*Department of Astronomy, UIUC, 1002 W Green Street, Urbana, IL 61801*

Big bang nucleosynthesis (BBN) and the cosmic microwave background (CMB) are two major pillars of cosmology. Standard BBN accurately predicts the primordial light element abundances (${}^4\text{He}$, D, ${}^3\text{He}$ and ${}^7\text{Li}$), depending on one parameter, the baryon density. Light element observations are used as a baryometer. The CMB anisotropies also contain information about the content of the universe which allows an important consistency check on the Big Bang model. In addition CMB observations now have sufficient accuracy to not only determine the total baryon density, but also resolve its principal constituents, H and ${}^4\text{He}$. We present a global analysis of all recent CMB data, with special emphasis on the concordance with BBN theory and light element observations. We find $\Omega_B h^2 = 0.0250^{+0.0019}_{-0.0026}$ and $Y_p = 0.250^{+0.010}_{-0.014}$ (fraction of baryon mass as ${}^4\text{He}$) using CMB data alone, in agreement with ${}^4\text{He}$ abundance observations. The determination of Y_p allows us to constrain the relativistic degrees of freedom during BBN, measured through the effective number of light neutrino species, $N_{\nu,eff} = 3.02^{+0.85}_{-0.79}$, in accord with the Standard Model of Particle physics. With this concordance established we show that the inclusion of standard, $N_{\nu,eff} \equiv 3$, BBN theory priors significantly reduces the volume of parameter space. In this case, we find $\Omega_B h^2 = 0.0245^{+0.0015}_{-0.0028}$ and $Y_p = 0.2493^{+0.0007}_{-0.0010}$. We also find that the inclusion of deuterium abundance observations reduces the Y_p and $\Omega_B h^2$ ranges by a factor of ~ 2 . Further light element observations and CMB anisotropy experiments will refine this concordance and sharpen BBN and the CMB as tools for precision cosmology.

I. INTRODUCTION

Big-bang nucleosynthesis (BBN) and the cosmic microwave background anisotropy (CMB) are two pillars of the hot big bang model.

The theory of BBN has long stood as an emblem of the predictive power of the Big Bang model [1, 2, 3, 4]. BBN has long provided the most reliable measurement of the cosmological baryon density. However, the CMB is rapidly becoming the preferred method for determining the baryon density, with its rapidly increasing precision. With the CMB, light element abundance observations become a powerful probe of the universe.

The observation and analysis of cosmic microwave background (CMB) anisotropies have attracted a great deal of attention in recent years due to their unique relevance for cosmological theory (see [5] for a recent review). A flood of observational results have been published during the last two years[6, 7, 8, 9, 10, 11, 12, 13]. These observations taken together measure CMB anisotropies over a large range of angular scales. The CMB is sensitive to the properties of the photon-baryon fluid and hence allows a precision determination of the baryon density at redshift $z \sim 1000$.

It is therefore apparent that combining BBN and CMB provides an opportunity for meaningful consistency

checks on the standard cosmology and has the potential to be powerfully predictive probe of nuclear and particle astrophysics at low and high redshift [14, 15, 16, 17, 18, 19, 20]. Given the constraint on the baryon density from the CMB, BBN yields a tight prediction of primordial ${}^4\text{He}$ abundance.

We also explore the promise of combining the CMB data with measurements of the deuterium (D/H) abundance, showing that current measurements of (D/H) can combine with the CMB constraints to reduce the error bars on ${}^4\text{He}$ by another factor of 2.

In section 2 we explain the data (CMB, BBN) and the method used to determine the likelihood surface in parameter space - the Metropolis-Hastings Markov Chain Monte Carlo (MCMC) algorithm. In section 3 the parameter confidence intervals that are extracted from that likelihood surface are discussed. In section 4 we discuss the implications of our results and how precision might be increased with further cosmological data.

II. METHODS

Our cosmological parameters are measured by use of the Markov Chain Metropolis Algorithm. We allow the ${}^4\text{He}$ mass fraction to float as an independent variable, yielding the following parameter space: $\Omega_m, \Omega_\Lambda, n_s, h, \Omega_B h^2, Y_p, \tau, r, n_t$ where $\Omega_{(m,\Lambda,B)}$ is the matter, cosmological constant, and baryon contents, respectively, Y_p is the fraction of the baryons in ${}^4\text{He}$ by mass, h is the Hubble constant in units of 100km/s Mpc , $n_{(s,t)}$ are the power-law index of the primordial scalar

*Electronic address: greg0huey@isildur.astro.uiuc.edu

†Electronic address: cyburt@astro.uiuc.edu

‡Electronic address: bwandelt@uiuc.edu

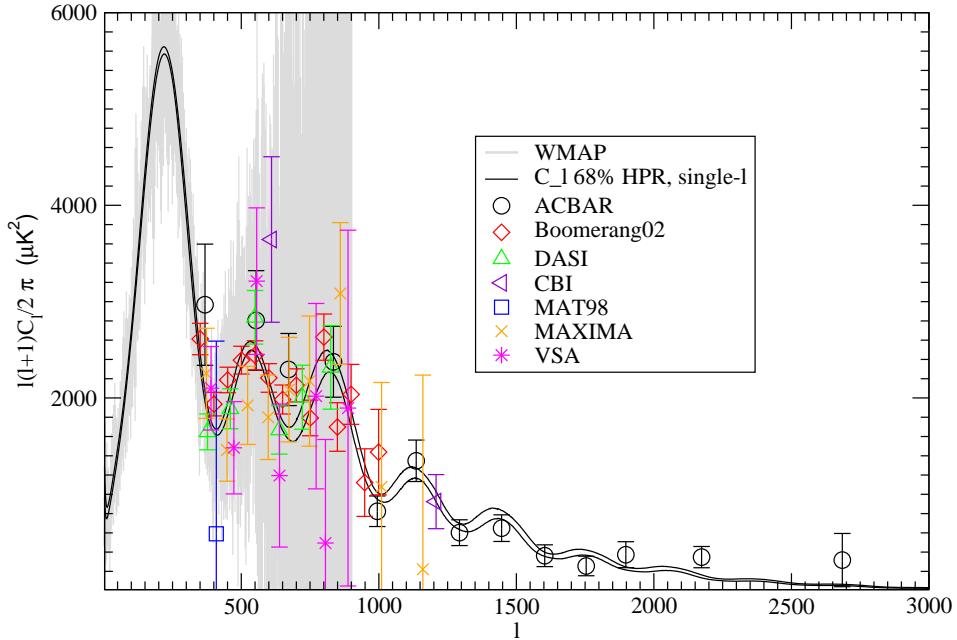


FIG. 1: The figure shows the data used in this paper (see legend) and 68% confidence intervals in the space of inferred theoretical power spectra (solid).

and tensor perturbations respectively, r is the fraction of the observed CMB quadrupole that is tensor and τ is the optical depth to the last-scattering surface (that is, re-scattering of CMB photons by reionization is allowed for). Note that in our selection of the parameter space simplifying assumptions have been made: adiabatic, scale-free primordial perturbations, the universe contains only cold dark matter and a cosmological constant and the neutrino species are strictly those of the Standard Model. To determine the region of this parameter space allowed by experimental data, one must sample the space over a wide range of points. At a given point the relative likelihood of the parameter values yielding the observations must be determined, and the range of points sampled must adequately cover the space. Our primary likelihood calculation is a comparison of the simulated CMB spectra produced by CMBFAST [21] against the WMAP CMB experiment [13], along with smaller-scale (bin $\ell_{eff} > 350$) data from the following experiments: Toco98 [6], DASI [7], Maxima [8], VSA [9], ACBAR [10], Boomerang02 [11], and CBI [12] [41]. We include the published calibration uncertainties for each experiment and find the maximum likelihood value for these parameters at each point in the cosmological parameter space.

Figure 1 shows the data we used as well as the most likely 68% of our inferred theory power spectra.

One could attempt to sample the parameter space on a uniform grid, but the high dimensionality coupled with the computational demands of CMBFAST makes this impossible in a reasonable amount of time. Instead, we implemented the Metropolis-Hastings MCMC Algorithm[22, 23, 24]: starting from the current point

in parameter space \vec{X}_i one proposes a test point \vec{p} , drawn from a distribution described by a density function called the proposal density: $P(\vec{p}|\vec{X}_i)$. The choice of proposal density is somewhat arbitrary, but a poor choice will cause the parameter estimation procedure to be inefficient. The likelihood relative to \vec{X}_i is computed at the test point \vec{p} . If the likelihood at \vec{p} is greater, then point $\vec{X}_{i+1} = \vec{p}$. Otherwise there is a probability that $\vec{X}_{i+1} = \vec{p}$ equal to the likelihood of \vec{p} divided by the likelihood of \vec{X}_i . If the proposal density is not symmetric in its arguments then this probability needs to be corrected by the factor $P(\vec{X}_i|\vec{p})/P(\vec{p}|\vec{X}_i)$ in order to enforce detailed balance. If \vec{X}_{i+1} is not set to the the point \vec{p} , then it is set to the current point: $\vec{X}_{i+1} = \vec{X}_i$. After a sufficient number of iterations the resulting density of points $\{\vec{X}_i | i = 1 \dots n\}$ asymptotically approaches the likelihood function on parameter space. What constitutes a “sufficient” number of points is in general difficult to determine (and impossible to determine with absolute certainty) - though there are tests which are good indicators. We use a test for our Markov Chain which was suggested by [25] and also used by the WMAP team[26].

We tune our Markov Chain code for optimal efficiency by first finding the maximal likelihood point in parameter space using a global maximization method (within our prior space, see below) then using it as the starting point for a sample Markov Chain. The variance of the sample chain is used to compute a step size matrix that will be used by the main chains. An efficient Markov Chain should take steps that are not too large or too small -

either will result in an inefficient, slowly converging chain.

We choose our proposal density to be a multivariate Gaussian, whose covariance matrix V_s is proportional to the covariance of the sample chain V_a . The proportionality constant $\alpha(D)$ is chosen so if the underlying distribution were Gaussian, 50% of the Markov chain points would be accepted. Its value depends on the number of dimensions of parameter space, D (for us $D = 9$). We found:

$$\alpha(D) \simeq 0.54784 D - 0.36159 \quad (1)$$

We ran two sets of Markov Chains, A and B , each consisting of 20 independent chains. Each chain was started at a point chosen from the distribution of the sample chain. Set A had only very weak top-hat priors ($\Omega_B h^2 \in [0.014, 0.030]$, $Y_p \in [0.13, 0.34]$, $h \in [0.45, 0.95]$, $\Omega_m \in [0.03, 1.00]$, $\Omega_\Lambda \in [0.00, 0.97]$, $n_s \in [0.5, 1.4]$, $n_t \in [-3.0, 3.0]$, $r \in [0.0, 3.0]$, $\tau \in [0.0, 1.0]$), whereas set B additionally had two strong priors: a BBN consistency condition between $\Omega_B h^2$ and Y_p [27], and the constraint $h = 0.74 + 0.11 - 0.094$ [28] [42]. The BBN consistency condition is simply this: for a given baryon density one expects BBN to produce a certain abundance of ${}^4\text{He}$, with some theoretical error (mostly driven by uncertainties in the nuclear cross-sections). In set A we treat $\Omega_B h^2$ and Y_p as two independent variables. In set B we enforce theoretical self-consistency between those variables. This non-CMB constraint is incorporated in a Bayesian way. The additional information is included as an additional prior and the density to be sampled from (the posterior density) becomes

$$\begin{aligned} P_{tot}(\vec{X}) &= P_{CMB}(\vec{X}) P_{BBN}(\Omega_B h^2, Y_p) P_{Hubble}(h) \\ &= P(\vec{X}|CMB) P(Y_p|\Omega_B h^2) P(h|Hubble\ data) \end{aligned} \quad (2)$$

where $P(x|y)$ is the conditional density for getting x , given y . Note that $P(Y_p|\Omega_B h^2)$ is purely a theoretical prior enforcing the BBN relation, and contains no abundance measurements. The Markov Chain then automatically explores the new, more constrained region of parameter space. As one might expect, sets A and B differ significantly in their parameter space coverage, and thus their proposal densities and chain starting points were determined separately.

Though a Markov Chain approach saves significant computational time, it is difficult to guarantee after some number of points that the chain has converged sufficiently to the true, underlying distribution. Indeed, a chain cannot tell one anything about a region it has not yet visited. We use a convergence test suggested by Gelman & Rubin [25], which was also employed by WMAP [26].

We have generalized this criterion to multiple dimensions, keeping in mind that any convergence test must be covariant (if a transformation of parameter space can change the determination of “convergence”, then the test is a bad one). Each chain out of the 20 has its own mean and variance [43]. If each chain reflected the underlying

distribution, then the variance of the means of the chains should be much less than the variance of the underlying distribution. We thus compute the variance of the chain means, multiply with the inverse of the average chain variance, and take the trace:

$$\begin{aligned} U &\equiv \frac{1}{N-1} \sum_{j=1}^N \left(\vec{\bar{X}}_j - \vec{\bar{X}} \right) \otimes \left(\vec{\bar{X}}_j - \vec{\bar{X}} \right) \\ W &\equiv \frac{1}{N} \sum_{j=1}^N \frac{1}{n_j-1} \sum_{i=1}^{n_j} \left(\vec{X}_{j,i} - \vec{\bar{X}}_j \right) \otimes \left(\vec{X}_{j,i} - \vec{\bar{X}}_j \right) \\ \mu &\equiv \text{Tr} [UW^{-1}] / D \end{aligned} \quad (3)$$

where N is the number of chains (20), n_j is the number of points in chain j , $\vec{\bar{X}}$ is the total mean, $\vec{\bar{X}}_j$ is the mean of chain j . We require that $\mu < 0.1$. Set B easily satisfies this criteria with 30,000 points, whereas set A required about 60,000 points. The average chain variance, W , is used because this underestimates the variance of the distribution until convergence is attained.

As a self-consistency check, one can take the point distribution of set A and combine it with the BBN $\Omega_B h^2 - Y_p$ and Hubble Key project priors. The resulting distribution should be the same as set B . The extent to which these distributions differ is a measure of non-convergence of the sets. We determined that the 68% confidence regions of these distributions more than 95% overlap in the $(\Omega_B h^2, Y_p)$ plane. A Markov Chain can be combined with a prior after the generation of the chain by assigning a weight to each point. The likelihood of a region in parameter space is then the weighted density of the points in that region. Because a Markov Chain maintains the full D -dimensional likelihood distribution in the parameter space, after it is generated the chain may be convolved with any arbitrary other likelihood function in that parameter space. Thus one can generate a Markov Chain distribution for WMAP alone, and chose any subset of the other cosmological datasets to convolve it with - for very little additional CPU cost. This is the basis of the *Cosmic Concordance Project* (CCP) website <http://galadriel.astro.uiuc.edu/ccp/> where the parameter constraints from chains A and B can be explored and combined with other cosmological datasets and priors. Further details about the CCP and about our parameter estimation methodology will be given in [29].

III. RESULTS

As our Markov Chain sets A and B approached 60,000 and 30,000 points respectively, convergence test eq. 3 gave $\mu \simeq 0.05 - 0.06$ [44] and we declared our chains sufficiently converged to provide reliable statistics. Figure 2a shows the 68% confidence region of set A in the $(\Omega_B h^2, Y_p)$ plane. Thus from CMB data alone we find $\Omega_B h^2 = 0.025^{+0.0019}_{-0.0026}$ and $Y_p = 0.250^{+0.010}_{-0.014}$

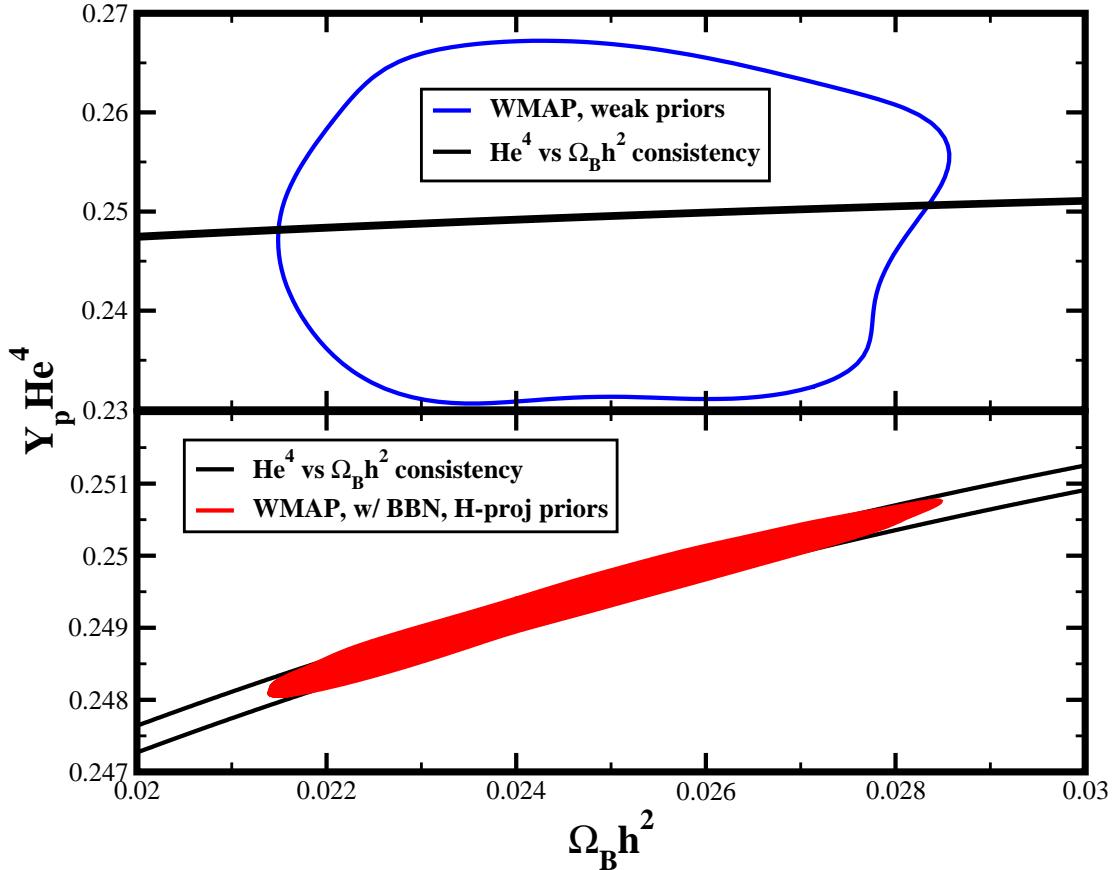


FIG. 2: The top figure shows the 68% confidence region in the $\Omega_B h^2$, Y_p plane from the CMB data alone (WMAP and high- ℓ data from other recent CMB experiments) bounded by the blue line. The black line bounds the 68% confidence region from BBN theory alone, using the best fit $\Omega_B h^2$ from the CMB. Note that the BBN theory band is in good agreement with the CMB data. The bottom figure shows the result of combining CMB and BBN data using an expanded Y_p axis. The solid red region is the 68% confidence region for the set B Markov chains which have as priors the BBN constraint and the Hubble Key project constraint on h . Note that the BBN constraint greatly reduces the allow range of Y_p as a function of $\Omega_B h^2$.

It is worth noting also that Markov chain set A yields a tight constraint on the neutrino number during BBN: Allowing the number of BBN neutrinos to float, one can find a 68% confidence interval for the number of BBN neutrinos needed to yield the determined Y_p from the determined $\Omega_B h^2$. Given a BBN code that computes the probability density $P(Y_p|\Omega_B h^2, N_{\nu,eff})$ (where the stochasticity is due to the measurement uncertainty in the relevant nuclear cross-sections) we can compute $P(N_{\nu,eff})$ based on CMB data and BBN theory as

$$P(N_{\nu,eff}|\text{BBN, CMB}) = \int dY_p d(\Omega_B h^2) P(N_{\nu,eff}|Y_p, \Omega_B h^2) P(Y_p, \Omega_B h^2) = \int dY_p d(\Omega_B h^2) P(Y_p|\Omega_B h^2, N_{\nu,eff}) P(Y_p, \Omega_B h^2).$$

The first term under each integral enforces the BBN relation and the second term the CMB pos-

terior. The second equality holds true if we assume flat priors $P(Y_p|\Omega_B h^2, BBN) = const.$ and $P(N_{\nu,eff}|\Omega_B h^2, BBN) = const$ for the range of parameter space of interest [45]. We find $N_{\nu,eff} = 3.02^{+0.85}_{-0.79}$. $N_{\nu,eff}$ is consistent with the standard model value of 3 and previous studies [15, 16, 17, 18, 19, 20].

However, $\Omega_B h^2$ and Y_p are jointly constrained by BBN theory, and thus are not really independent variables. Adopting the standard BBN model ($N_{\nu,eff}(\text{BBN})=3$) of [27] yields a consistency relation between $\Omega_B h^2$ and Y_p . The 68% confidence region of this consistency relation appears in Figure 2a as a narrow band (narrow enough that the upper and lower bounding curves appear to merge). Enforcing this condition greatly increases the precision of parameter estimation, as evident in table 1, with dramatic affect on Y_p measurement: $Y_p = 0.2493^{+0.0007}_{-0.0010}$. This is simply a result of the accuracy with which ${}^4\text{He}$ is determined by BBN ($\sim 0.1\%$). In Figure 2b we have zoomed in on this CMB-BBN con-

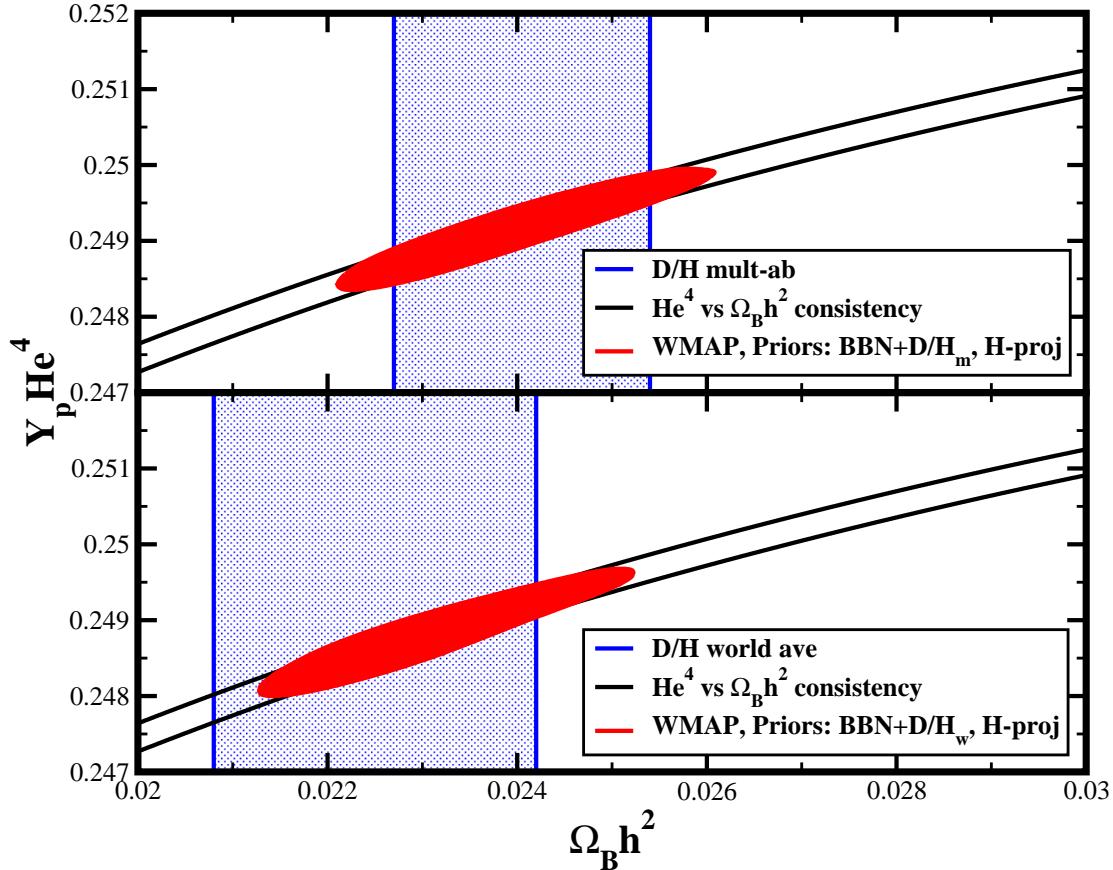


FIG. 3: The primordial abundance of deuterium can be used to further constrain the baryon density (blue shaded band is 68%), which in turn increases the precision of the Y_p determination (solid red region is 68%). Two different D abundances are used: 2a) data from 2 multiple-line absorption systems, and 2b) data from 5 systems (the 2 multiple-line plus 3 others) (see text).

cordance region. Also shown is the 68% confidence region of set B shaded as solid red. As one would expect, set B agrees with the product of the CMB (set A) and BBN (consistency band) likelihoods. It is important to note that in Figure 2a the agreement between the CMB and BBN allowed regions need not have happened. Instead, the BBN consistency band might not have passed through the high CMB likelihood region, which would have forced one to consider a BBN scenario other than the standard model one with 3 neutrinos. The CMB-BBN agreement reaffirms the standard BBN scenario.

Model A and Model B compare quite well to ${}^4\text{He}$ observations. Olive, Skillman and Steigman (1997) [30] and Fields and Olive (1998) [31] find $Y_p = 0.238 \pm 0.002$, while Izotov and Thuau (1998) [32] find $Y_p = 0.244 \pm 0.002$. The errors cited are statistical only. Comparing these numbers, not only are they discrepant from each other, but they lie below the mean value determined in this evaluation. However, Olive and Skillman (2001) [33] critically evaluate the methods used in determining Y_p and find a lower bound to a systematic error, $\sigma_{sys} \geq 0.005$.

This systematic error is added in quadrature with the statistical error to determine the total error, increasing the errors to 0.0054. The Y_p observations are brought into marginal accord with each other and the CMB; both lie systematically lower than the CMB determined value. As discussed earlier, the systematic error used is only a lower bound, and as such the true errors are most likely larger than those quoted.

In Figure 2 the allowed $\Omega_B h^2$ is very large. Any other data that can reduce the allowed $\Omega_B h^2$ range will have the additional benefit of refining the precision of the Y_p measurement. As an example, we consider D abundance in Figure 3. The value of D/H is still a somewhat open question due to low number statistics, and thus we demonstrate the effects of two different D abundances. In Figure 3a we use the average of the 2 multiple absorption line systems of [34, 35]: $D/H = (2.49 \pm 0.18) \times 10^{-5}$. The blue shaded band is the 68% confidence baryon density range allowed by this D/H value as determined with the BBN theory of [27]. The solid red region is the 68% confidence region resulted from the convolution of the D

TABLE I: Cosmological Parameter Estimates

Parameters	Model A	Model B
$\Omega_B h^2$	$0.0250^{+0.0019}_{-0.0026}$	$0.0245^{+0.0015}_{-0.0029}$
Y_p	$0.250^{+0.010}_{-0.014}$	$0.2493^{+0.0007}_{-0.0010}$
h	$0.684^{+0.057}_{-0.097}$	0.733 ± 0.059
Ω_M	$0.241^{+0.062}_{-0.064}$	$0.219^{+0.041}_{-0.058}$
Ω_Λ	$0.792^{+0.063}_{-0.047}$	$0.798^{+0.060}_{-0.045}$
n_s	$1.047^{+0.062}_{-0.075}$	$1.024^{+0.041}_{-0.086}$
r	$0.168^{+0.065}_{-0.141}$	$0.117^{+0.084}_{-0.117}$
n_t	$0.089^{+0.456}_{-0.258}$	$0.207^{+0.408}_{-0.347}$
τ	$0.228^{+0.103}_{-0.123}$	$0.180^{+0.058}_{-0.127}$

baryon range and the concordance region of Figure 2b. Thus we find $Y_p = 0.2491^{+0.0004}_{-0.0005}$, $\Omega_B h^2 = 0.0237^{+0.0010}_{-0.0012}$. Alternatively, one can use a conservative combination of 5 D/H measurements, including the two multiple absorption line systems employed above [34, 35, 36, 37, 38]: $D/H = (2.78 \pm 0.29) \times 10^{-5}$ (the overall error increase is because the other three systems are not consistent with each other or the multiple absorption line systems, a hint of an underlying systematic error for the single absorption line systems). For this D abundance we find $Y_p = 0.2488^{+0.0004}_{-0.0005}$ and $\Omega_B h^2 = 0.0230^{+0.0008}_{-0.0012}$.

IV. CONCLUSIONS

This work has been based on two general ideas; (1) BBN and the CMB independently probe two different epochs, providing valuable consistency checks for the underpinnings of the Standard Cosmological Model; (2) having established that the cosmological model agrees remarkably well with these very different observational probes we use these data to make a precision measurement of the ^4He abundance. We have presented an analysis of all recent CMB data, in which we have determined the cosmic baryon density and the primordial helium abundance. We found $\Omega_B h^2 = 0.0250^{+0.0019}_{-0.0026}$ and $Y_p = 0.250^{+0.010}_{-0.014}$ at 68% from CMB data alone. This is consistent with 3 standard model neutrinos during BBN.

We have shown that this is fully consistent with the predicted ^4He abundance from BBN theory, and

marginally consistent with ^4He observations. The likely source of this slight discrepancy is an underestimate of the dominant, systematic uncertainties in the ^4He observations, which now seems affirmed with the CMB determination of Y_p . The agreement between the CMB-only set A confidence region of $\Omega_B h^2$, Y_p and the consistency band based on BBN theory shown in Figure 2a reaffirms the standard BBN model. Thus using BBN theory, we can effectively remove Y_p as a freely floating variable, enforcing the $\Omega_B h^2$ - Y_p BBN relation in CMB data analysis. Given this, we found the incorporation of BBN theory into parameter extraction from CMB data results in a precision measurement of the ^4He abundance. We find $\Omega_B h^2 = 0.0245^{+0.0015}_{-0.0029}$ and $Y_p = 0.2493^{+0.0007}_{-0.0010}$.

Using this CMB-BBN determined Y_p , one can study and constrain stellar evolution [39, 40]. One can similarly study the nucleosynthetic history of all the light element abundances as discussed in [16] and references therein.

We show the promise of incorporating deuterium abundance observations, yielding $Y_p = 0.2491^{+0.0004}_{-0.0005}$, $\Omega_B h^2 = 0.0237^{+0.0010}_{-0.0012}$ or $Y_p = 0.2488^{+0.0004}_{-0.0005}$, $\Omega_B h^2 = 0.0230^{+0.0008}_{-0.0012}$ depending on which systems are used to measure the deuterium abundance.

The addition of the D abundance observations are only one example of many possible cosmological datasets that might be incorporated into parameter extraction to increase precision. An experiment may constrain a parameter directly, or may reduce degeneracy in a related parameter. For example, using large-scale structure information to reduce the residual n_s - $\Omega_B h^2$ degeneracy in the current CMB data would also increase the precision of the Y_p determination. Also, further light element observations and CMB anisotropy experiments will refine this concordance and sharpen BBN and the CMB as tools for precision cosmology. Due to the effect of the ^4He abundance on the damping tail, this may improve the constraint on a possible running of scalar spectral index. These are the topics of on-going work and can be further explored at the *Cosmic Concordance Project* web-site: <http://galadriel.astro.uiuc.edu/ccp/> [29].

Many thanks to Prof. Brian Fields for helpful discussions. The work of R.H.C was provided by the National Science Foundation grant AST-0092939. This work was partially supported by National Computational Science Alliance under grant AST020003N and utilized the NCSA Platinum system.

[1] Walker, T.P. *et al.* 1991 ApJ 376, 51
[2] Sarkar, S. 1996 Rep. Prog. Phys. 59, 1493
[3] Olive, K.A., Steigman, G. & Walker, T.P. 2000 Phys. Rep. 333, 389
[4] Hagiwara, K. *et al.* 2002 Phys. Rev D 66, 010001; Section 19. “Big-bang Nucleosynthesis” Fields, B.D. & Sarkar, S.
[5] Dodelson, S. and Wayne, H., 2002, Ann. Rev. Astr. As-
troph. 40, 171-216
[6] Miller, A.D. *et al.* 1999 ApJ 524 (1999) L1-L4
[7] Halverson, N.W. *et al.* 2001, ApJ 568, 38 [arXiv: astro-ph/0104489]
[8] Hanany, S. *et al.* ApJ 545 (2000) L5
[9] Grainge, K. *et al.* Mon. Not. R. Astron. Soc. 000, 15 (2002)

- [10] Kuo, C. L. *et al.* 2002, ApJ, [arXiv: astro-ph/0212289]
- [11] Ruhl, J.E. *et al.* 2002, [arXiv: astro-ph/0212229]
- [12] Padin, S. *et. al.*, ApJ 549, L1, (2001)
- [13] Bennett, C.L. *et al.* 2003, accepted by ApJ, [arXiv: astro-ph/0302207]
- [14] Schramm, D.N. & Turner, M.S. 1998 Rev. Mod. Phys. 70, 303
- [15] Cyburt, R.H., Fields, B.D. & Olive, K.A. 2002 Astropart. Phys. 17, 87 [arXiv:astro-ph/0105397]
- [16] Cyburt, R.H. Fields, B.D. & Olive 2003 Phys. Lett. B 567, 227 [arXiv:astro-ph/0302431]
- [17] Barger, V. *et al.* 2003 Phys. Lett. B 566, 8 [arXiv:hep-ph/0305075]
- [18] Hansen,S.H. *et al.* 2002 Phys. Rev. D 65, 023511 [arXiv:astro-ph/0105385]
- [19] Kneller, J.P. *et al.* 2001 Phys. Rev. D 64, 123506 [arXiv:astro-ph/0101386]
- [20] Di Bari, P. & Foot, R. 2001 Phys. Rev. D 63, 043008 [arXiv:hep-ph/0008258]
- [21] Seljak, U. & Zaldarriaga, M. 1996, ApJ, 469, 437
- [22] Metropolis, N., *et al.* 1953 J. Chem. Phys. 21, 1087-1092
- [23] Christensen, N., *et al.* 2001 Classical and Quantum Gravity 18, 2677
- [24] Lewis, A. and Bridle, S., Phys.Rev. D66, 103511(2002)
- [25] Gelman, A. & Rubin, D., 1992 Statistical Science, 7, 457
- [26] Verde, L. *et al* 2003, accepted by ApJ., [arXiv: astro-ph/0302218]
- [27] Cyburt, R.H., Fields, B.D. & Olive 2001 New Astron. 6 215 [arXiv:astro-ph/0102179]
- [28] Freedman, W. L., *et al.* 2001, ApJ, 553, 47
- [29] Huey, G., Wandelt, B, Cyburt, R., in preparation (2003)
- [30] Olive, K.A., Skillman, E. & Steigman, G. 1997 ApJ 483, 788 [arXiv:astro-ph/9611166]
- [31] Fields, B.D. & Olive, K.A. 1998 ApJ 506, 177 [arXiv:astro-ph/9803297]
- [32] Izotov, Y.I. & Thuan, T.X. 1998 ApJ 500, 188
- [33] Olive, K.A & Skillman, E. 2001 New Astron. 6, 246 [arXiv:astro-ph/0007081]
- [34] O'Meara, J.M. *et al.* 2001 ApJ 552, 718 [arXiv:astro-ph/0011179]
- [35] Kirkman, D. *et al.* 2003 submitted to ApJS [arXiv:astro-ph/0302006]
- [36] Burles, S. and Tytler, D. 1998 ApJ 499, 699
- [37] Burles, S. and Tytler, D. 1998 ApJ 507, 732
- [38] Pettini, M. & Bowen, D. 2001 ApJ 560, 41 [arXiv:astro-ph/0104474]
- [39] Bono, G. *et al.* 2002 ApJ 568, 463 [arXiv:astro-ph/0112064]
- [40] Cassisi, S., Salaris, M. & Irwin, A.W. 2003 ApJ 588, 862 [arXiv:astro-ph/0301378]
- [41] For WMAP, the published likelihood function was used. For the other experiments BJK formalism was used, with the BJK parameters obtained from either RADPACK, or the collaborations directly.
- [42] The strong prior $h = 0.74 + 0.11 - 0.094$ is an asymmetric Gaussian with plus and minus sigmas as given, and represents the constraint coming from the Hubble key project [28]
- [43] For $D > 1$ we use the word variance to mean covariance matrix.
- [44] For this value we concatenated 5 chains into 1, thus transforming 20 chains per set to 4.
- [45] For simplicity, we approximate $P(Y_p, \Omega_B h^2 | CMB) = P(Y_p | CMB)P(\Omega_B h^2 | CMB)$ for this calculation. We also assume that the CMB does not constrain $N_{\nu,eff}$ (instead, we treat the effective number of neutrino species that are relevant for the CMB anisotropy as independent of $N_{\nu,eff}$) and that there is no *a priori* preference for any value of $N_{\nu,eff}$ or Y_p for any value of $\Omega_B h^2$.